Estimating Distinguishability Measures on a Quantum Computer

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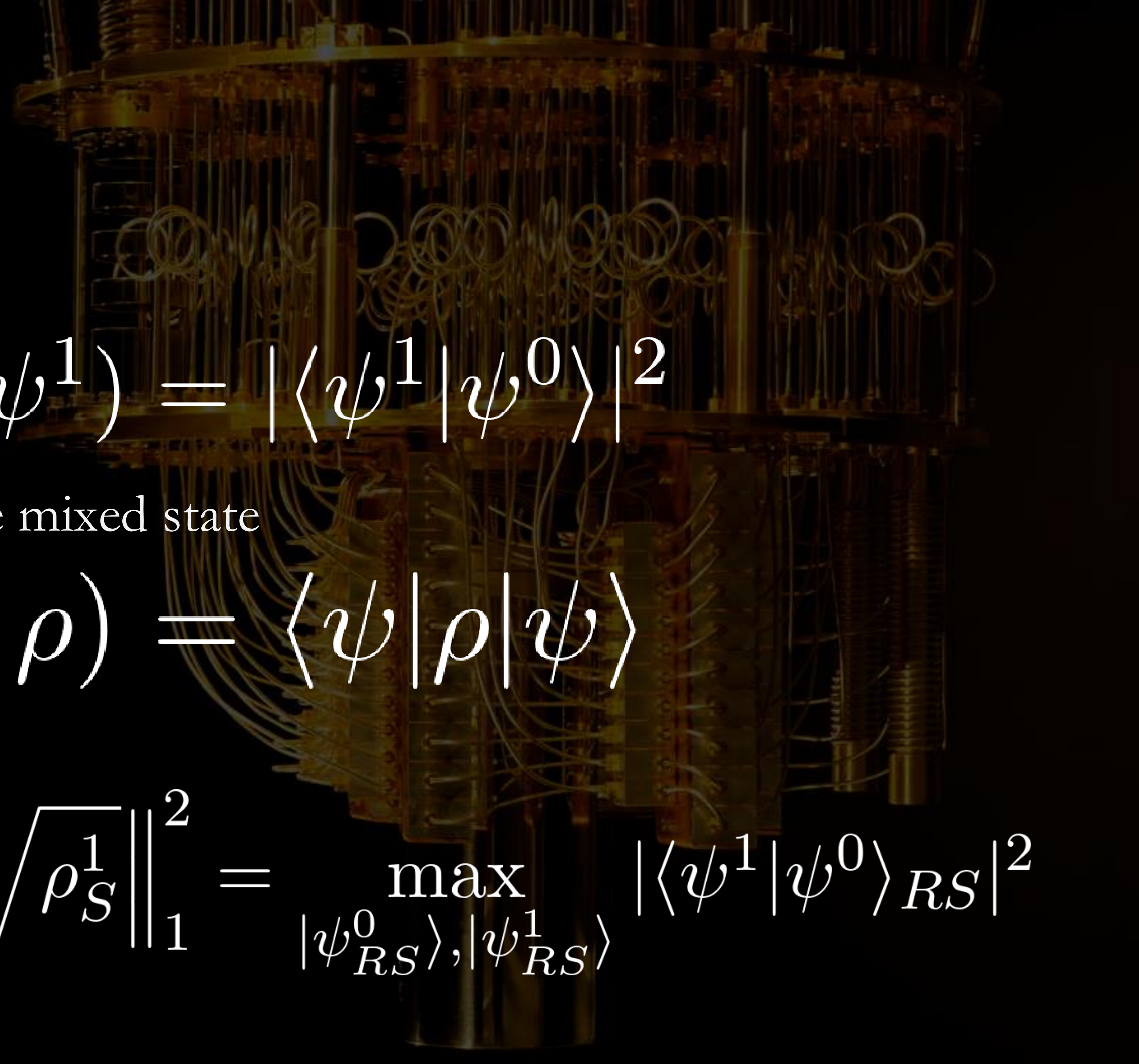
Introduction

- Measuring performance of any protocol relies on distinguishing the protocol from the ideal case.
- Commonly employed distinguishability measures - **trace distance** and **fidelity.**
- Generalizations to quantum channels and strategies.
- Efficiently computable by semi-definite programming.



Fidelity

- Fidelity of two pure states

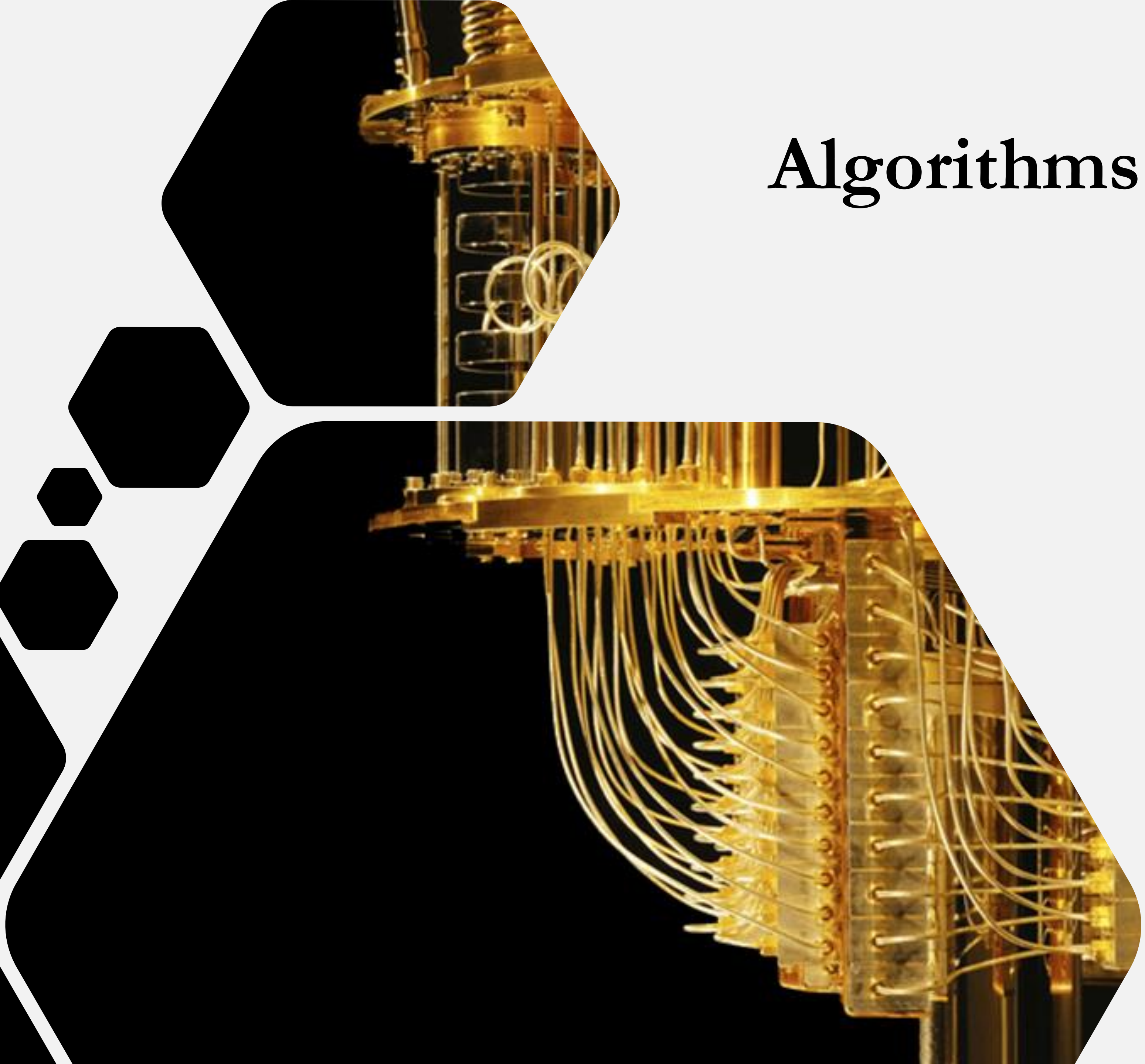
$$F(\psi^0, \psi^1) = |\langle \psi^1 | \psi^0 \rangle|^2$$

- Fidelity of one pure and one mixed state

$$F(\psi, \rho) = \langle \psi | \rho | \psi \rangle$$

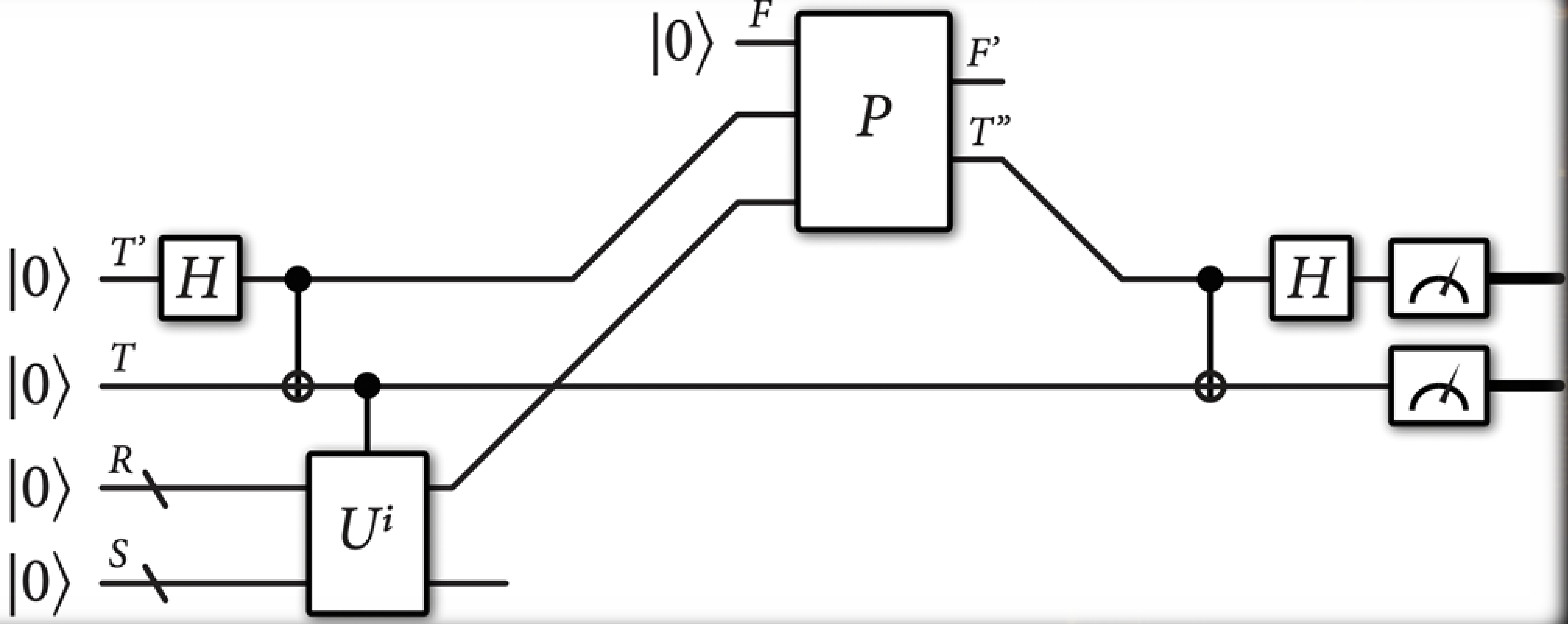
- Fidelity of two mixed states

$$F(\rho_S^0, \rho_S^1) = \left\| \sqrt{\rho_S^0} \sqrt{\rho_S^1} \right\|_1^2 = \max_{|\psi_{RS}^0\rangle, |\psi_{RS}^1\rangle} |\langle \psi^1 | \psi^0 \rangle_{RS}|^2$$

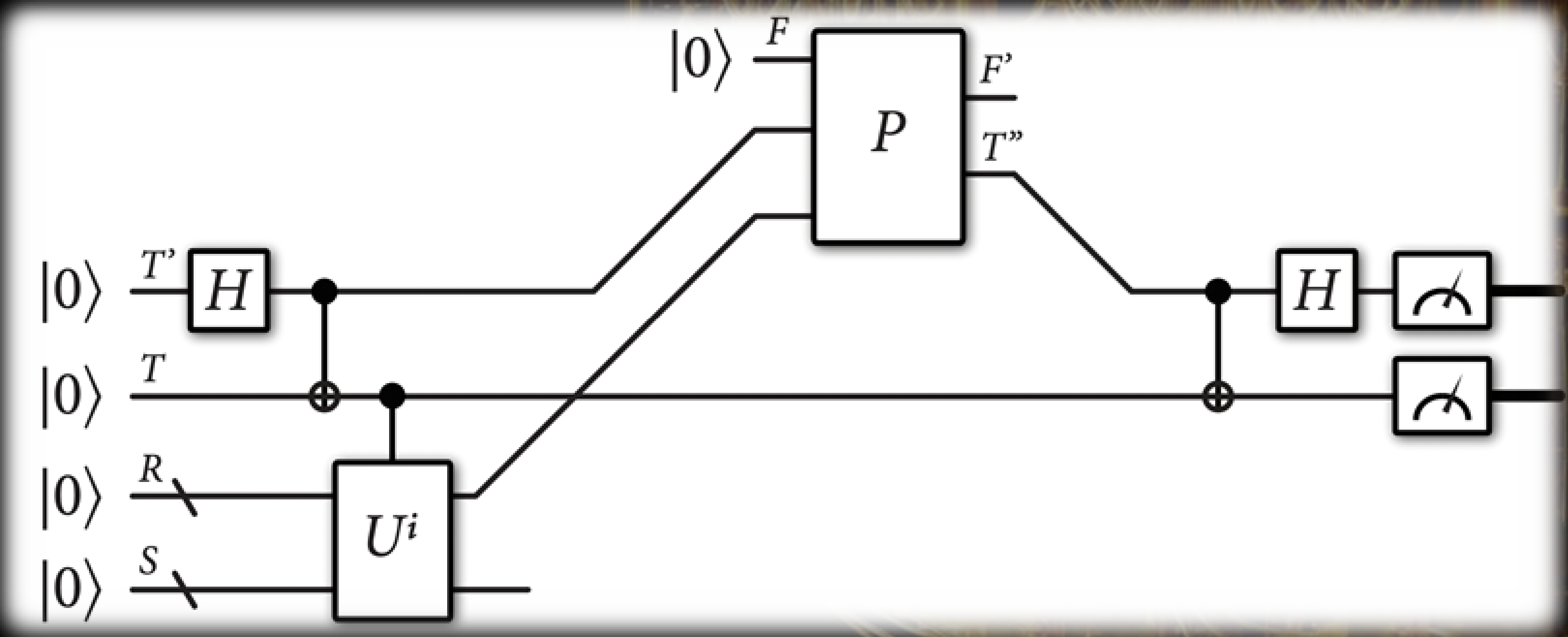


Algorithms to Estimate Fidelity

Algorithm 1

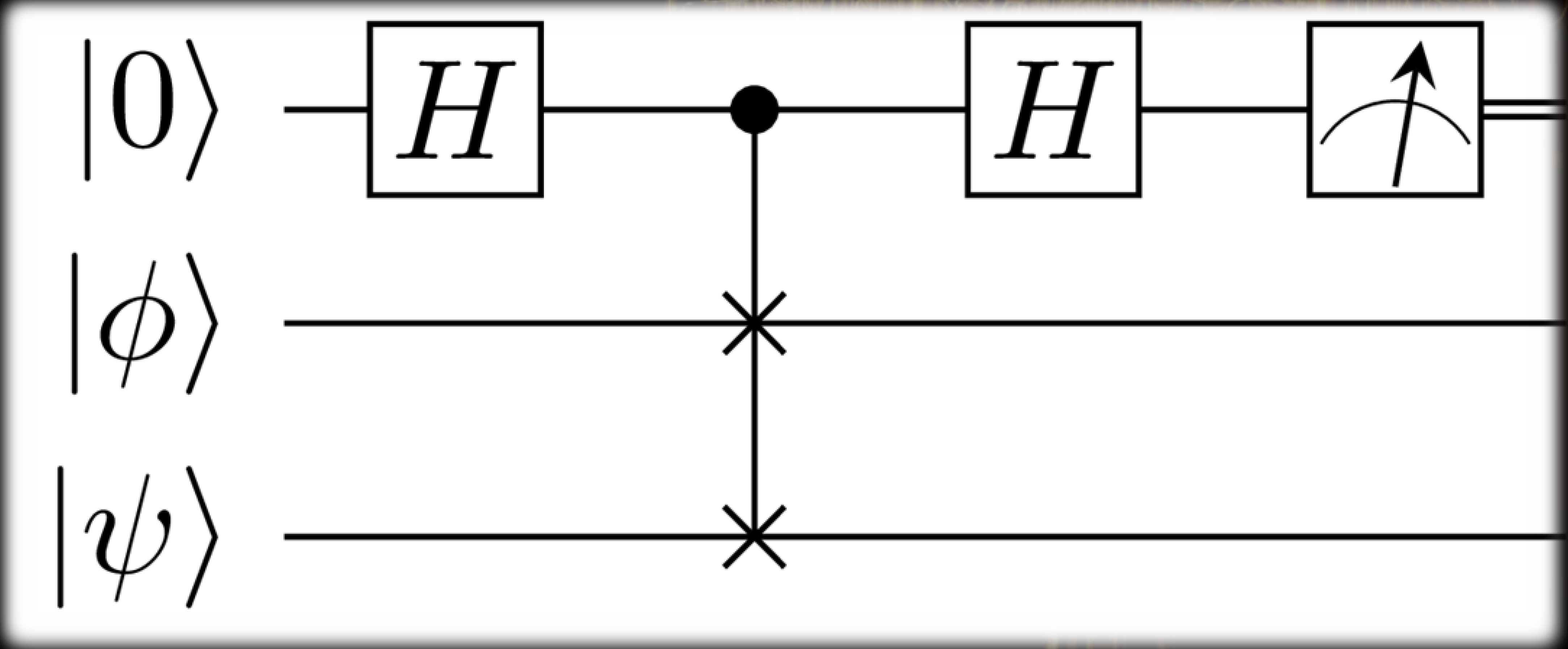


Algorithm 1

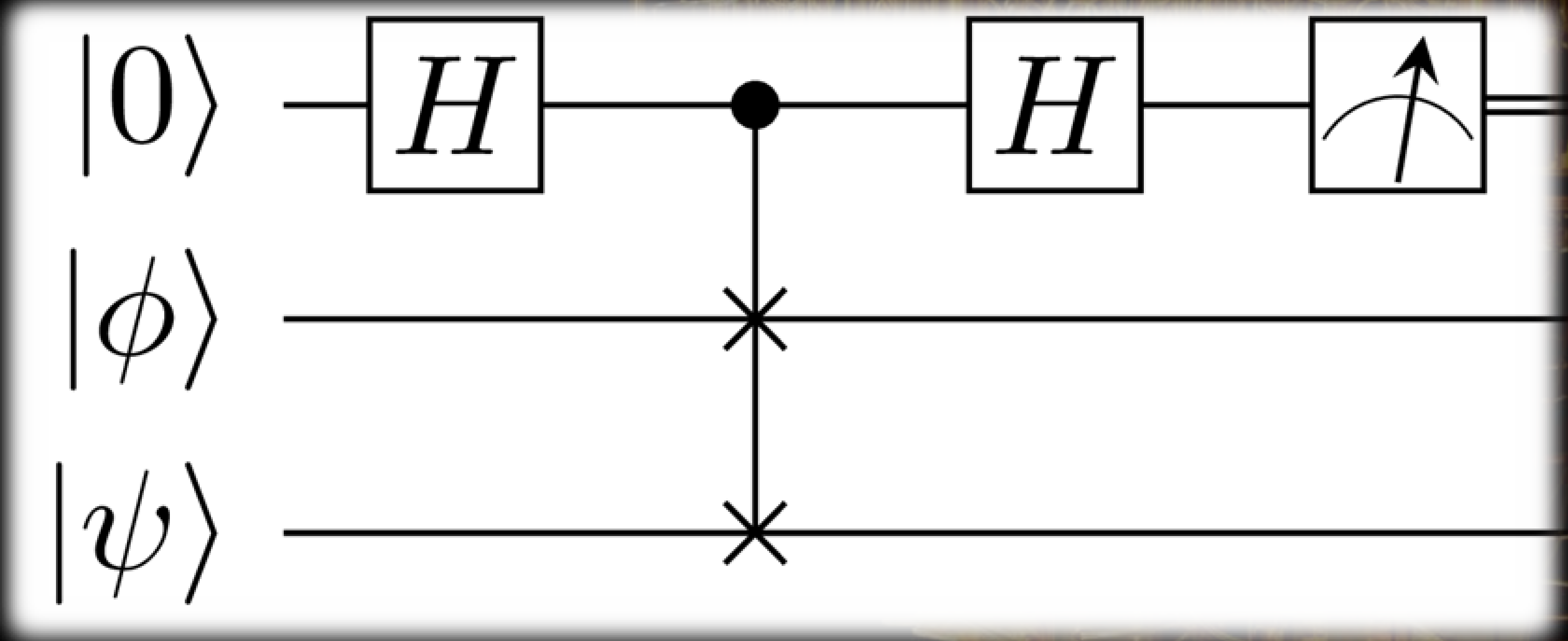


$$p_{\text{acc}} = \frac{1}{2}\left(1 + \sqrt{F}(\rho_S^0, \rho_S^1)\right)$$

Algorithm 2 – Swap Test

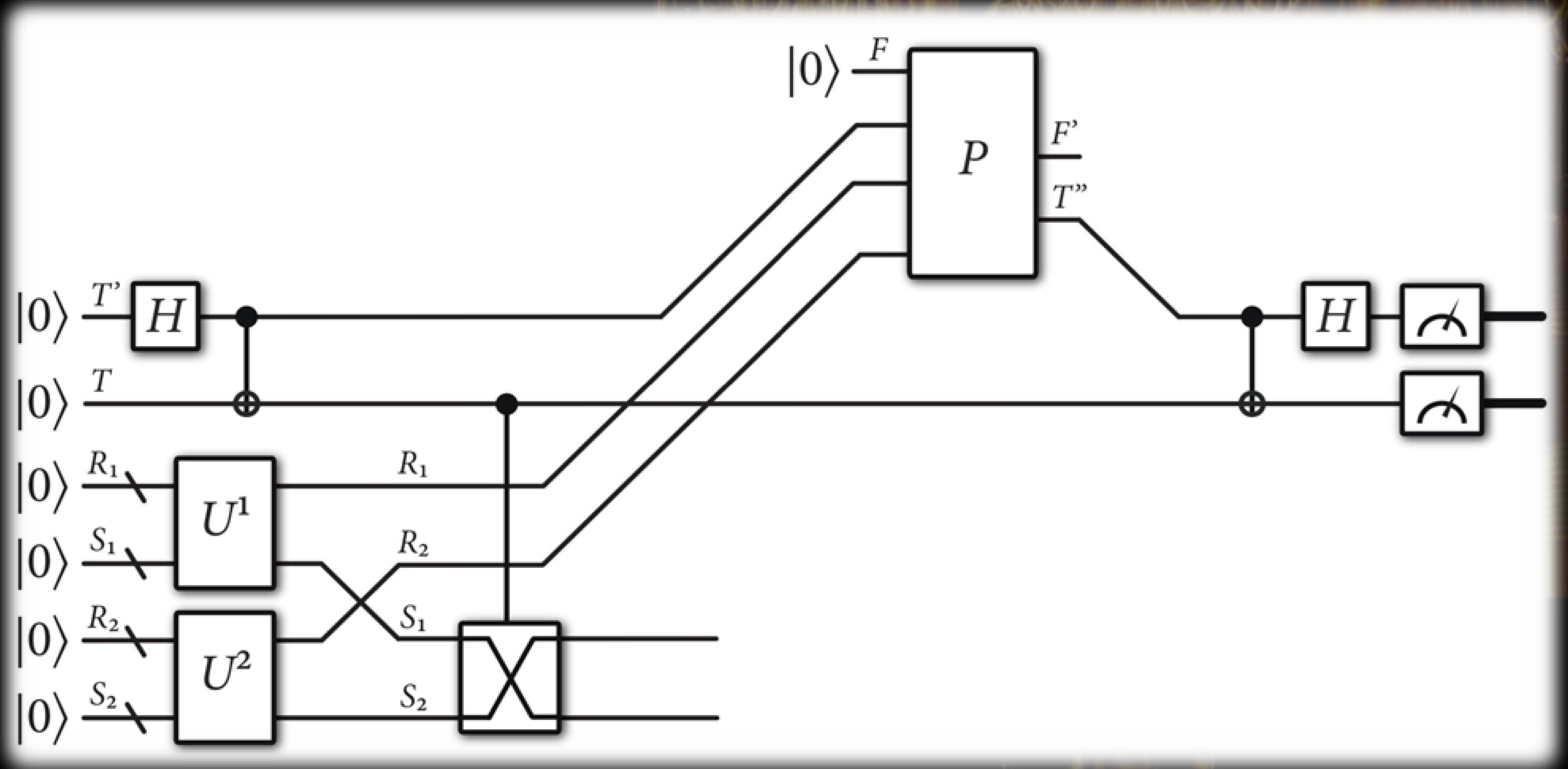


Algorithm 2 – Swap Test

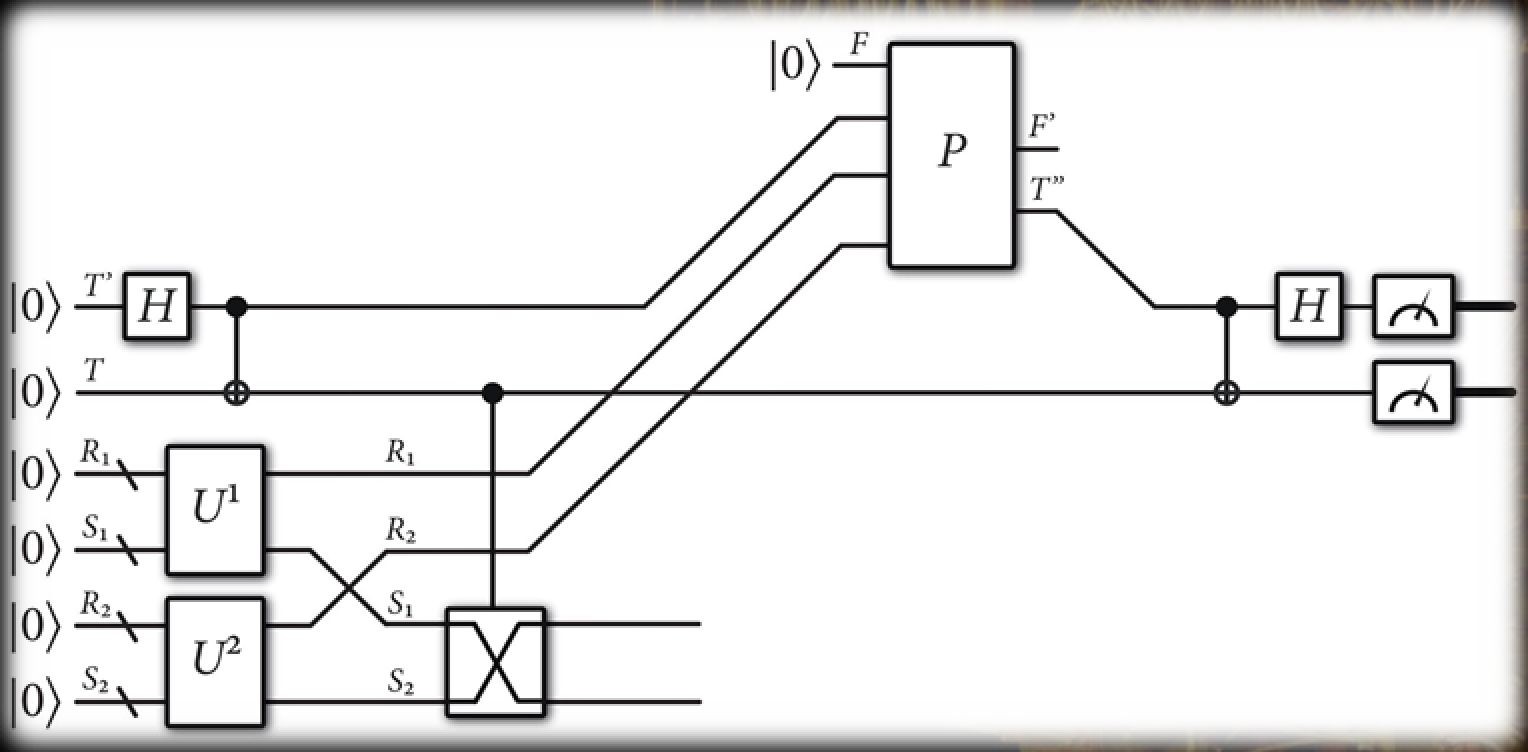


$$p_{\mathrm{acc}} = \frac{1}{2} + \frac{1}{2}|\langle\psi|\phi\rangle|^2$$

Algorithm 3 – Generalized Swap Test



Algorithm 3 – Generalized Swap Test



$$p_{\mathrm{acc}} = \frac{1}{2}\left(1 + F(\rho_S^0, \rho_S^1)\right)$$

Algorithm 4 – Bell Measurements

Let there be two 1 qubit pure states of a system S.

$$\mathrm{Tr}[SWAP\left(\psi_S \otimes \varphi_{\tilde{S}}\right)] = |\langle\psi|\varphi\rangle|^2 = F(\psi_S, \varphi_S)$$

$$SWAP = \Phi^+ + \Phi^- + \Psi^+ - \Psi^-$$

$$F(\psi_S, \varphi_S) = \mathrm{Tr}[(\Phi^+ + \Phi^- + \Psi^+ - \Psi^-)(\psi_S \otimes \varphi_{\tilde{S}})]$$

Algorithm 4 – Bell Measurements

To generalize to multi-qubit mixed states,

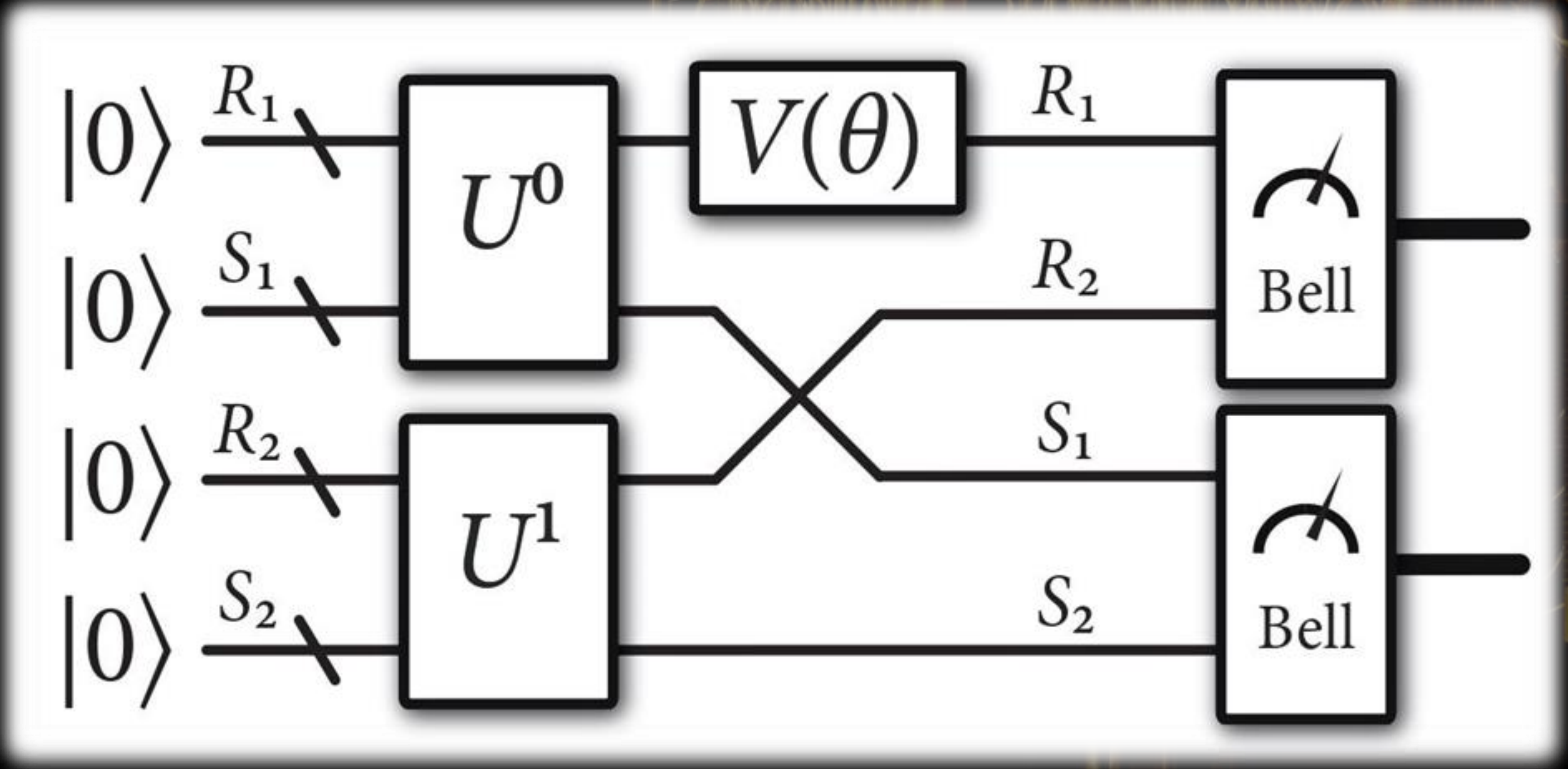
$$\text{Reward Function} = \overline{Y^n}(\theta) = \frac{1}{n}\sum_{j=1}^{n} Y_j(\theta)$$

$$Y_j(\theta) = (-1)^{\sum_{i=1}^{m} x_R^i \cdot z_R^i + x_S^i \cdot z_S^i}$$

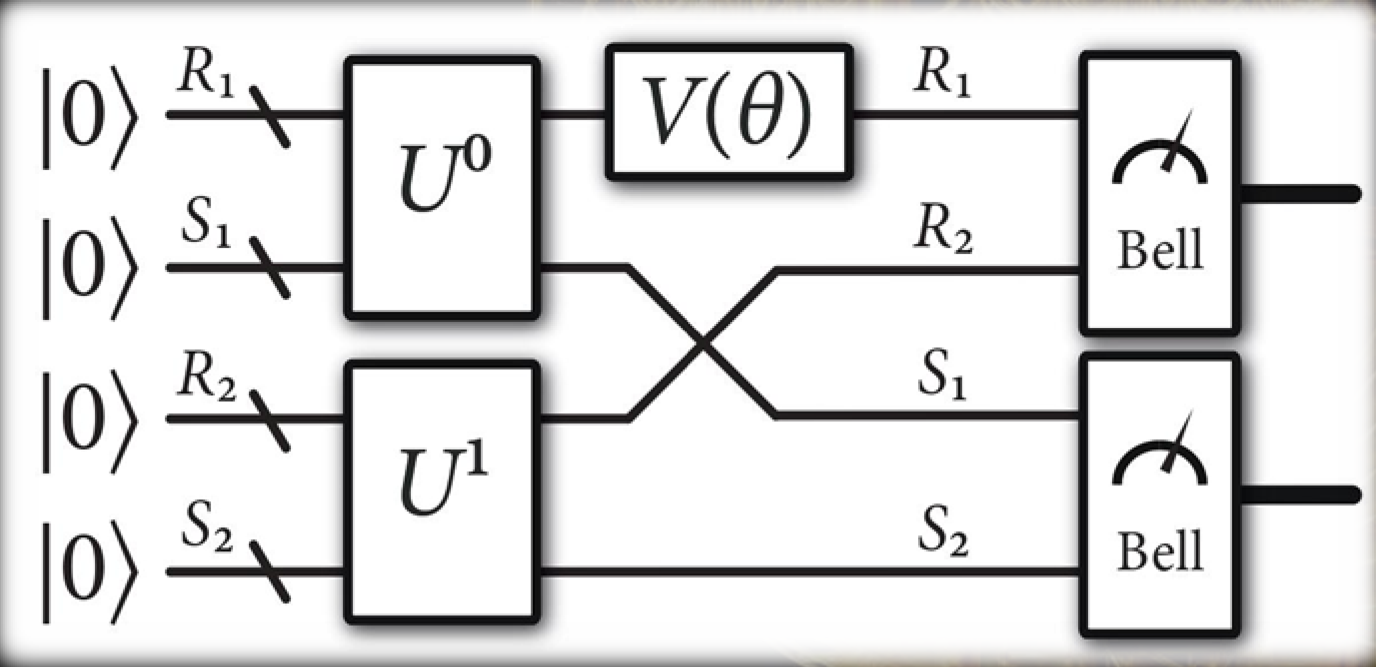
$$F_\theta \equiv \left| \langle \psi^{\rho^1} |_{RS} V_R(\theta) \otimes I_S | \psi^{\rho^0} \rangle_{RS} \right|^2$$

$$\left| \langle \psi^{\rho^1} |_{RS} V_R(\theta) \otimes I_S | \psi^{\rho^0} \rangle_{RS} \right|^2 = F(\psi_{RS}^{\rho^1}, V_R(\theta) \psi_{RS}^{\rho^0} V_R^\dagger(\theta))$$

Algorithm 4 – Bell Measurements



Algorithm 4 – Bell Measurements



$$p_{\mathrm{acc}} = \frac{1}{2}\left(1 + \sqrt{F}(\rho_S^0, \rho_S^1)\right)$$

Algorithm 5 – Fuchs Caves

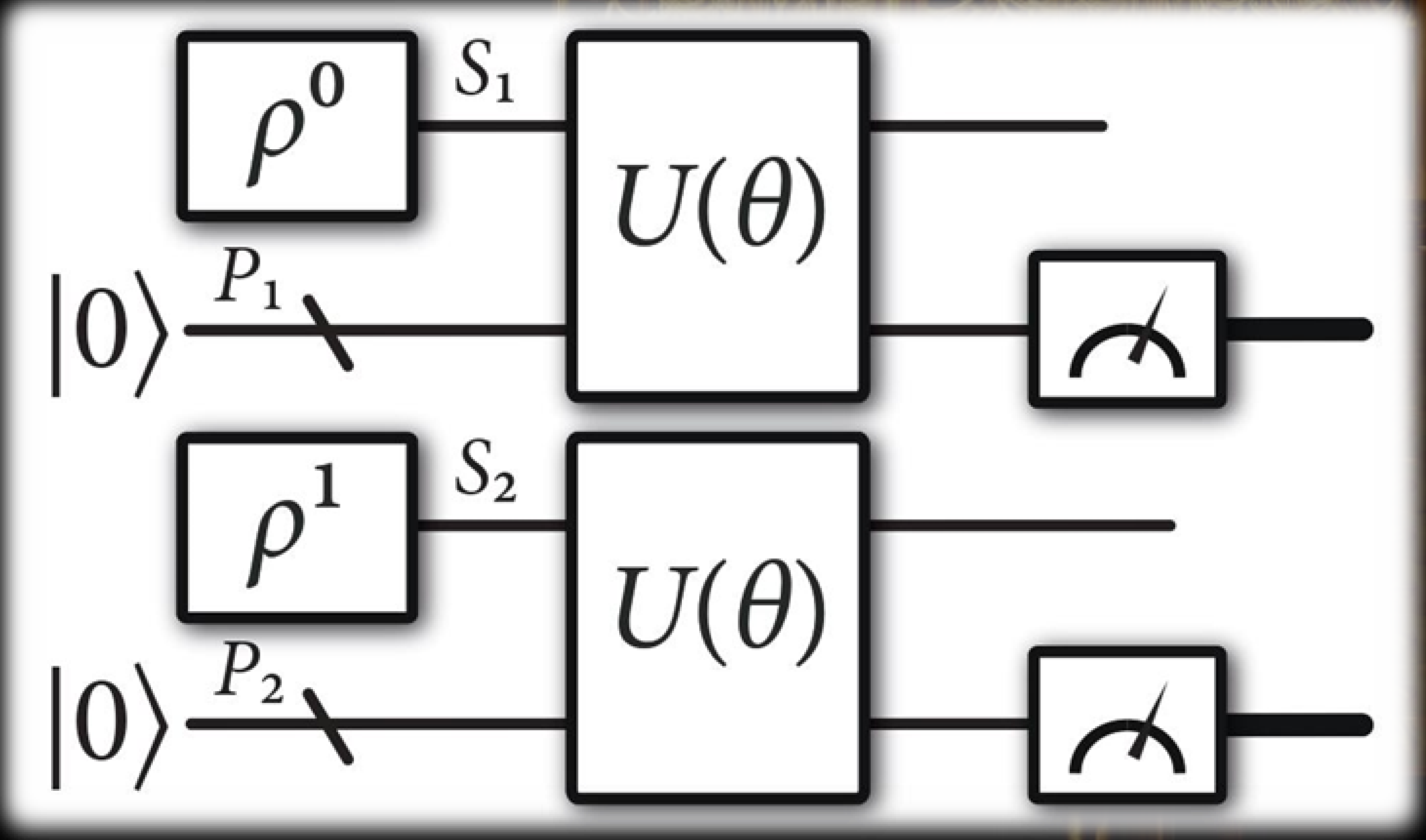
- Alternate expression optimizing over measurements

$$F(\rho_S^0, \rho_S^1) = \left[\min_{\{\Lambda_S^x\}_x} \sum_x \sqrt{\mathrm{Tr}[\Lambda_S^x \rho_S^0]\,\mathrm{Tr}[\Lambda_S^x \rho_S^1]} \right]^2$$

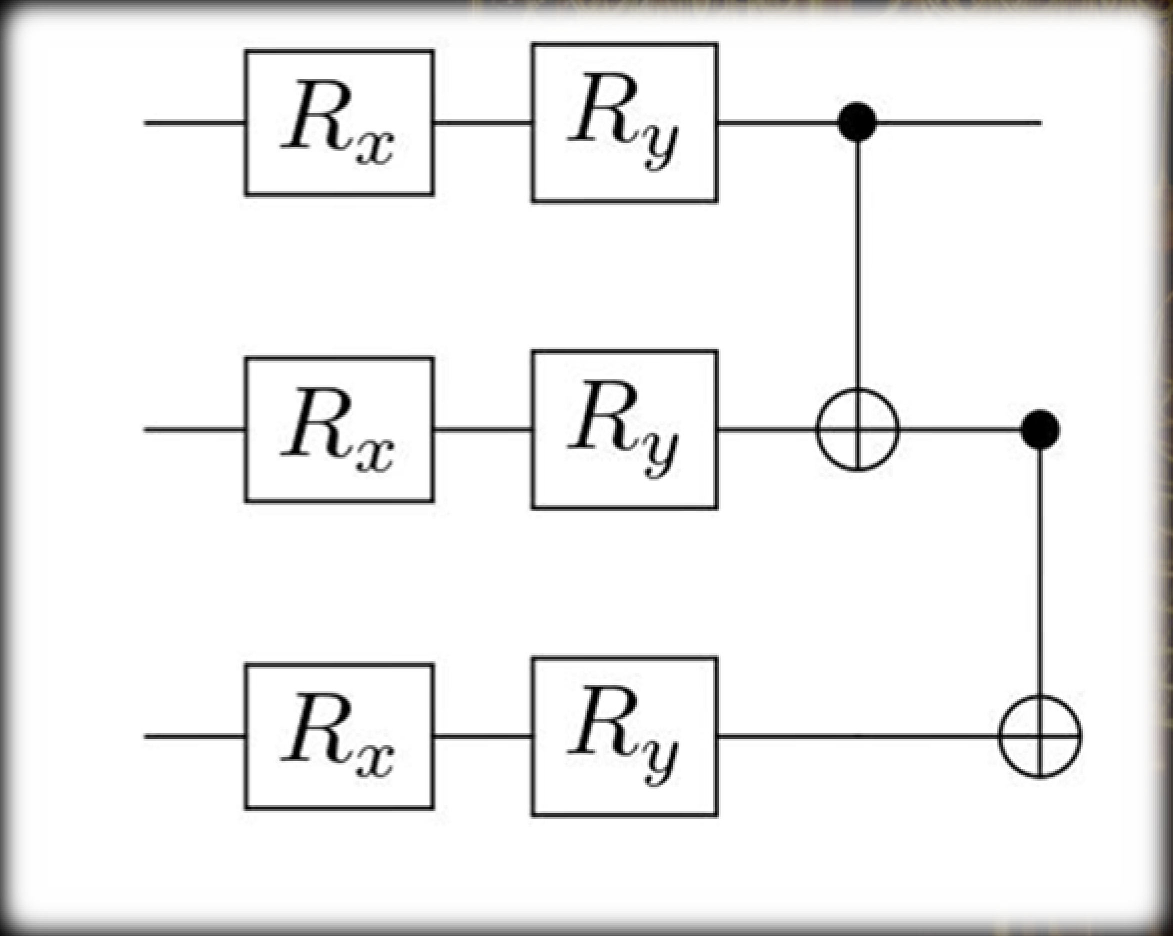
- From Naimark Extension Theorem,

$$\mathrm{Tr}[\Lambda_S^x \rho_S] = \mathrm{Tr}[(I_S \otimes |x\rangle\langle x|_P) U_{SP} (\rho_S \otimes |0\rangle\langle 0|_P) U_{SP}^\dagger]$$

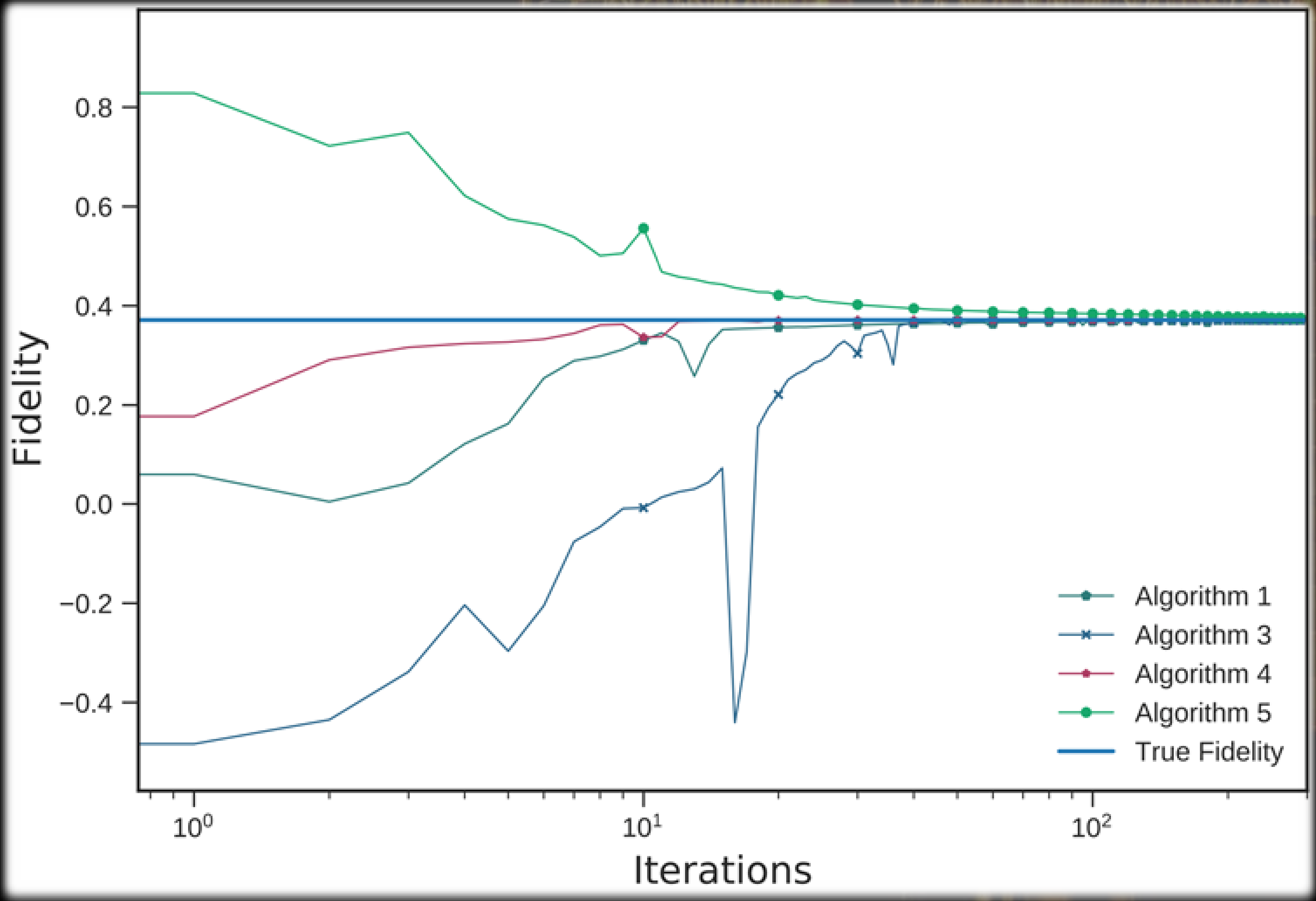
Algorithm 5 – Fuchs Caves



Simulations - Ansatz



Results

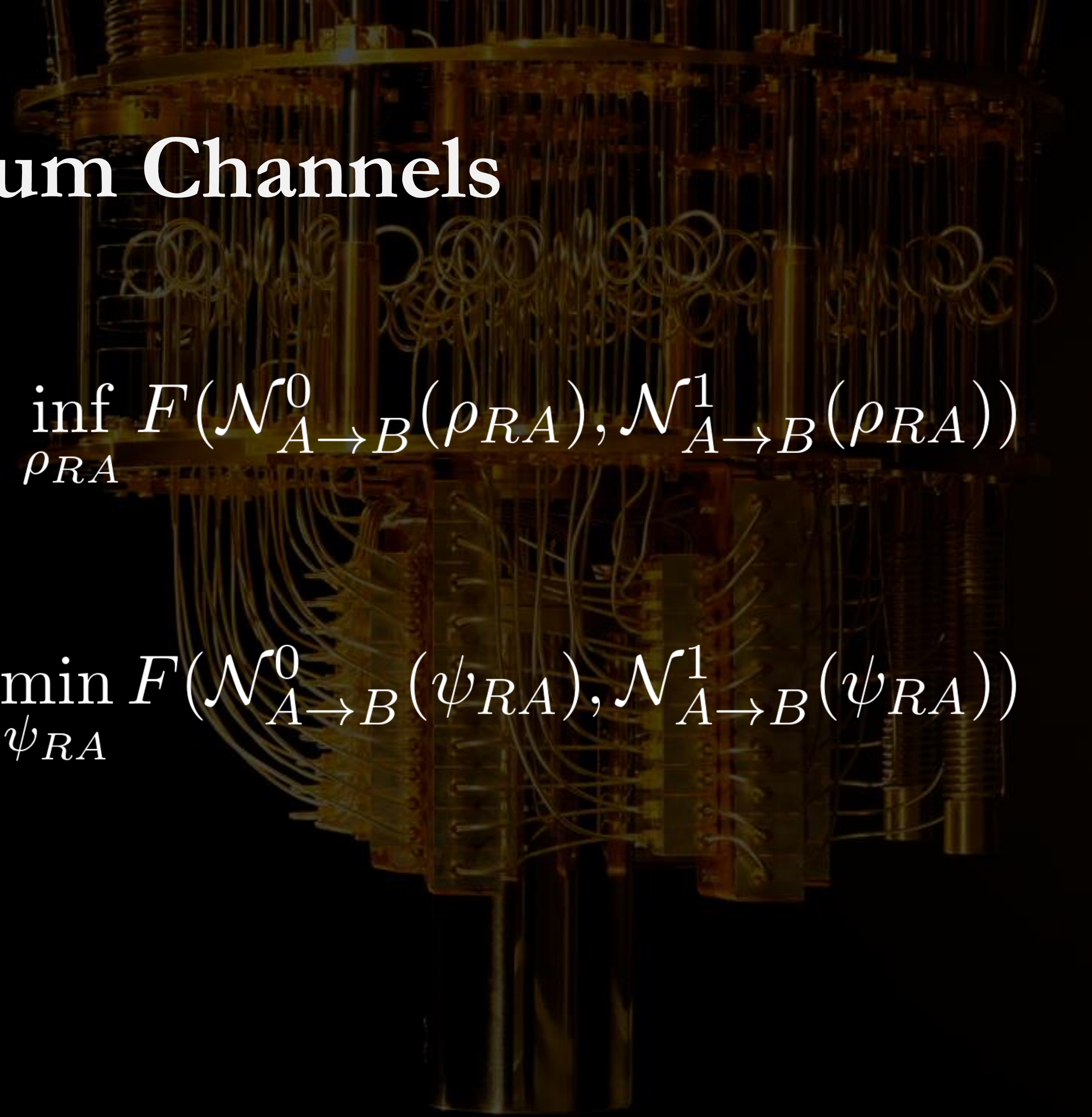


Fidelity
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
10^0
10^1
10^2
Iterations
Algorithm 1
Algorithm 3
Algorithm 4
Algorithm 5
True Fidelity

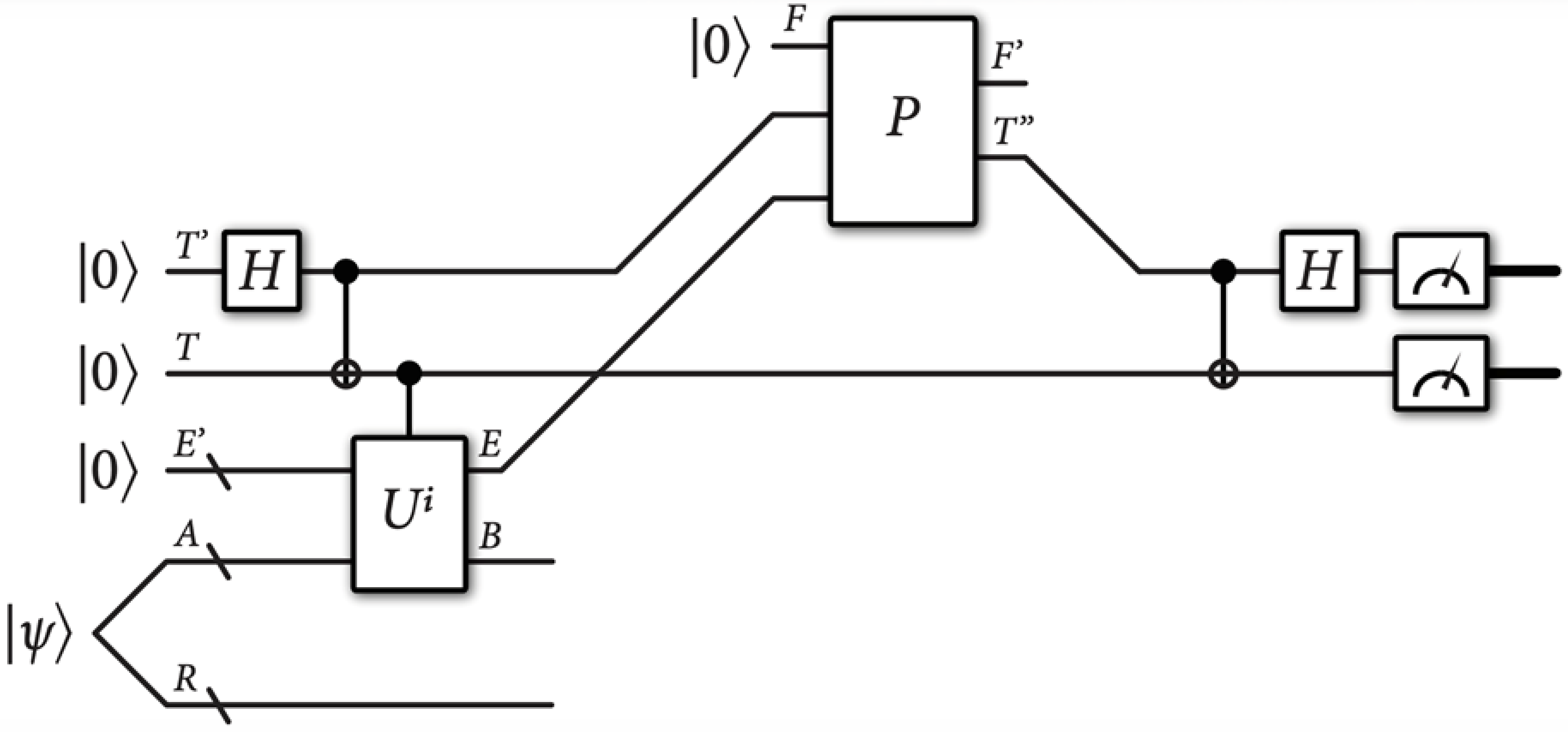
Fidelity of Quantum Channels

$$F(\mathcal{N}^0_{A\to B}, \mathcal{N}^1_{A\to B}) = \inf_{\rho_{RA}} F(\mathcal{N}^0_{A\to B}(\rho_{RA}), \mathcal{N}^1_{A\to B}(\rho_{RA}))$$

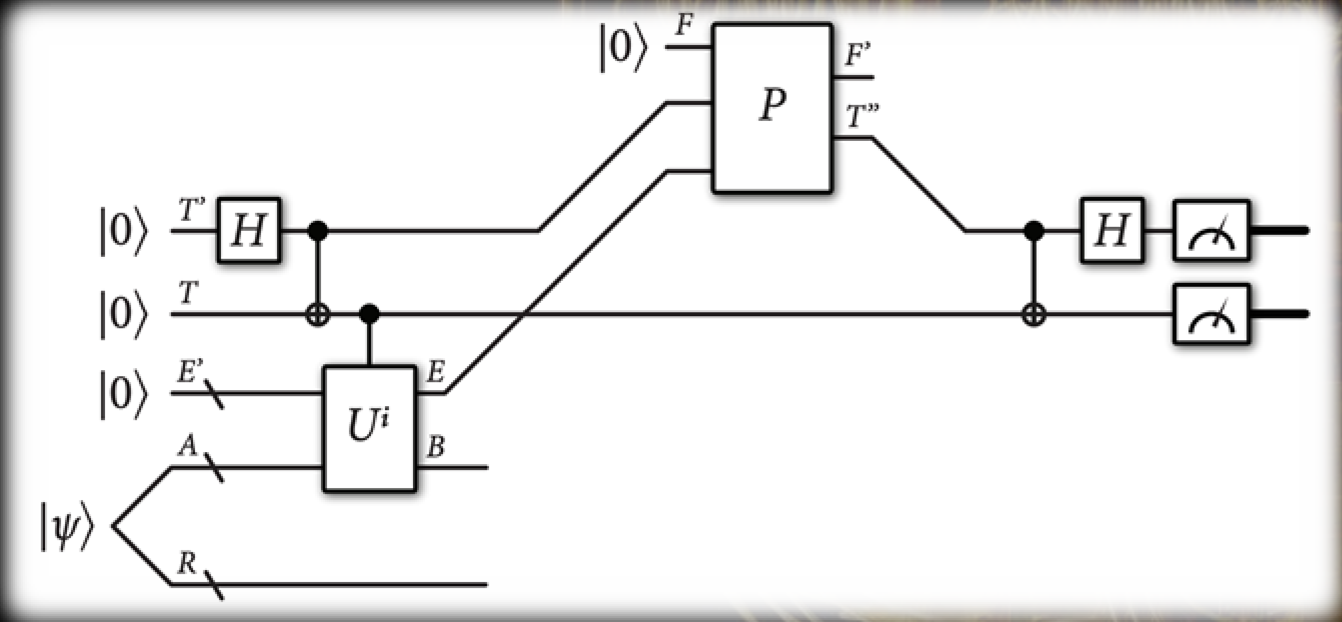
$$F(\mathcal{N}^0_{A\to B}, \mathcal{N}^1_{A\to B}) = \min_{\psi_{RA}} F(\mathcal{N}^0_{A\to B}(\psi_{RA}), \mathcal{N}^1_{A\to B}(\psi_{RA}))$$



Algorithm – Fidelity of Channels



Algorithm – Fidelity of Channels

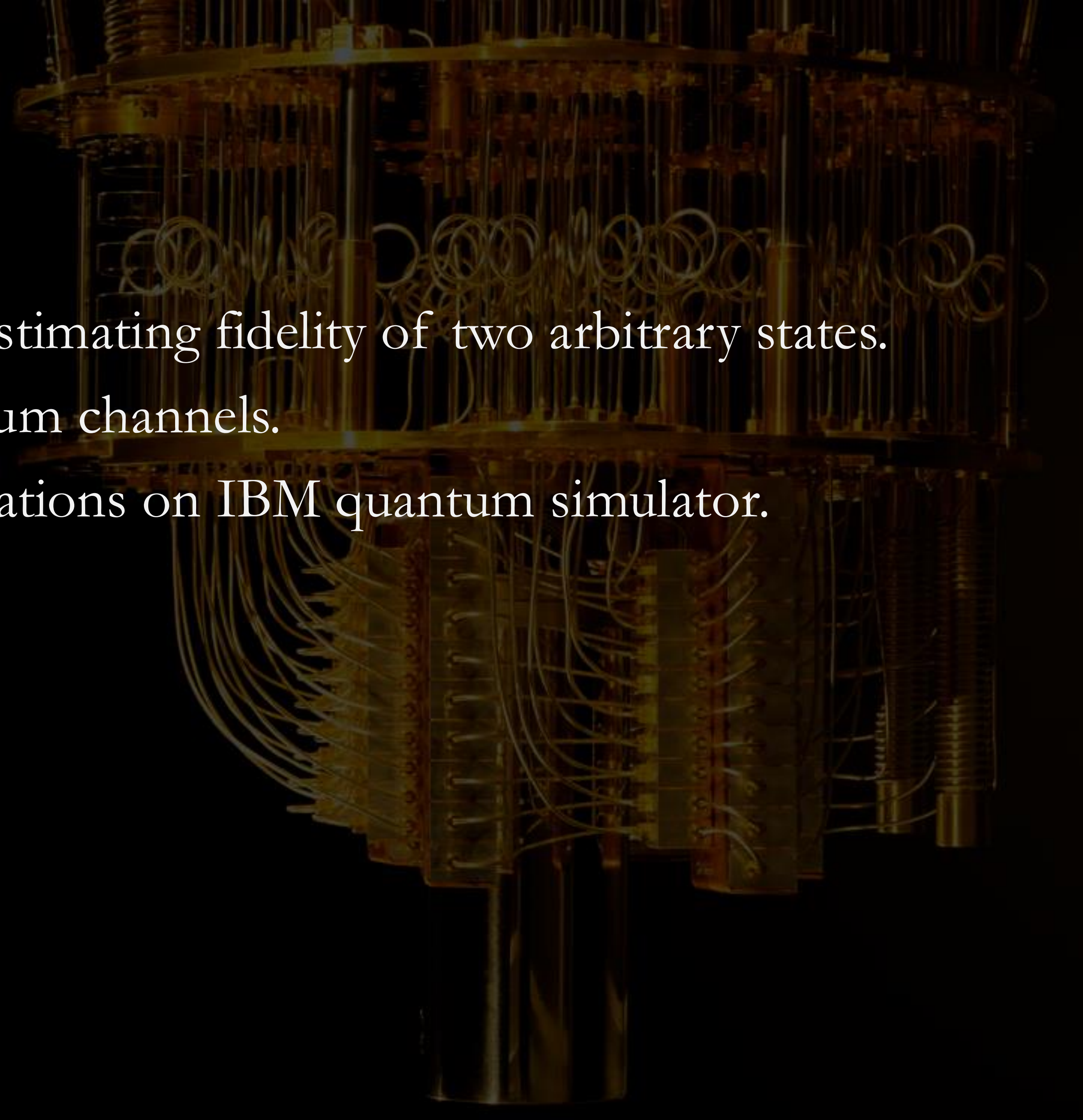


$$p_{\text{acc}} = \min_{|\psi\rangle_{RA}} \max_{P} \frac{1}{2} \left\| \langle \Phi|_{T''T} P \sum_{i \in \{0,1\}} |ii\rangle_{T'T} U^i |\psi\rangle_{RA} |00\rangle_{E'F} \right\|_2^2$$

$$= \frac{1}{2}\left(1 + \sqrt{F}\left(\mathcal{N}^0_{A\to B}, \mathcal{N}^1_{A\to B}\right)\right)$$

Conclusion

- We give four algorithms for estimating fidelity of two arbitrary states.
- They are extendable to quantum channels.
- We show the numerical simulations on IBM quantum simulator.



Open Questions

- Identifying conditions under which quantum computer can identify these quantities effectively.
- Effect of noise on these algorithms.
- Scalability of these algorithms.

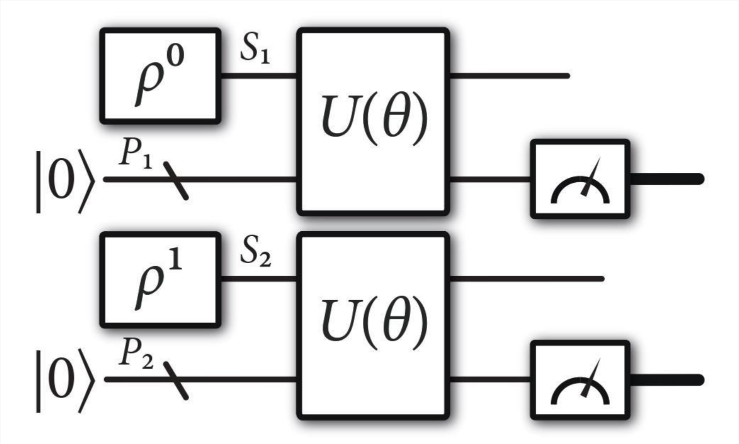
Thank You

Trace distance

The trace distance of two quantum states ρ_S^0 and ρ_S^1 is defined as

$$\left\|\rho_S^0 - \rho_S^1\right\|_1 \tag{1}$$

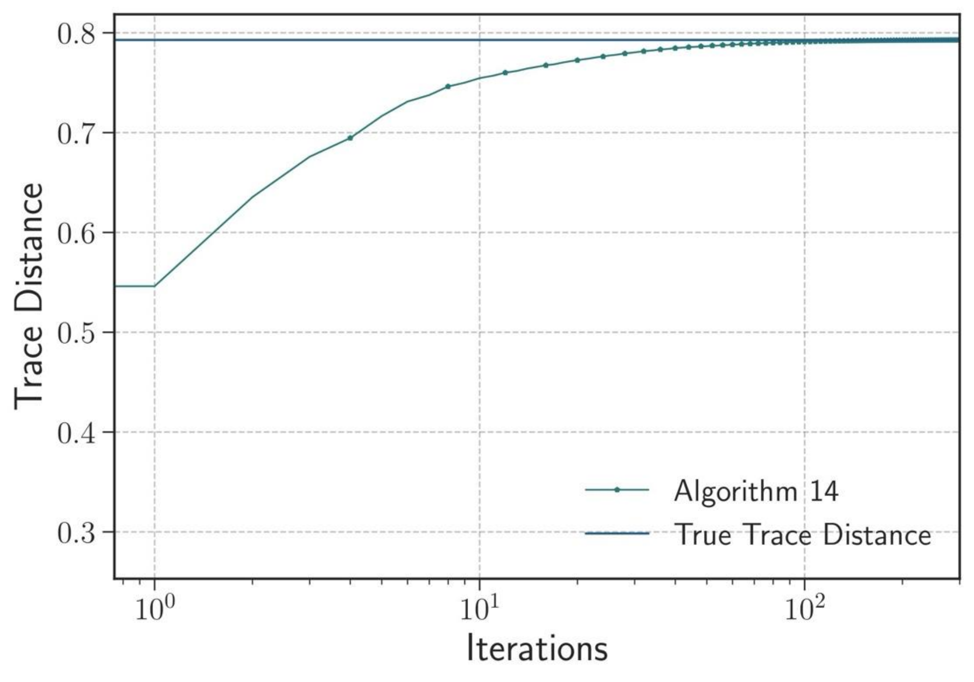
where $\|A\|_1 = \mathrm{Tr}[\sqrt{A^\dagger A}]$.



The quantum circuit to estimate trace distance. The two qubits are measured and the verifier accepts if the measurement outcome is 0 on first qubit and 1 on second qubit.

$$\begin{aligned} p_{\mathrm{acc}} &= \max_{\Lambda:0\le\Lambda\le I} \frac{1}{2}\mathrm{Tr}[\Lambda\rho_S^0] + \frac{1}{2}\mathrm{Tr}[(I-\Lambda)\rho_S^1] \\ &= \frac{1}{2}\left(1 + \frac{1}{2}\left\|\rho_S^0 - \rho_S^1\right\|_1\right) \end{aligned}$$

Simulations



We simulate the algorithms on local machines with no noise and find that the algorithm converges to the known fidelity with an absolute error of 10^{-5} in 300 iterations.

Diamond Distance

The diamond distance between quantum channels $\mathcal{N}^0_{A\to B}$ and $\mathcal{N}^1_{A\to B}$ is defined as

$$\left\|\mathcal{N}^0_{A\to B} - \mathcal{N}^1_{A\to B}\right\|_\diamond = \sup_{\rho_{RA}} \left\|\mathcal{N}^0_{A\to B}(\rho_{RA}) - \mathcal{N}^1_{A\to B}(\rho_{RA})\right\|_1, \quad (1)$$

where the optimization is over every bipartite state ρ_{RA} and the system R can be arbitrarily large. Also,

$$\left\|\mathcal{N}^0_{A\to B} - \mathcal{N}^1_{A\to B}\right\|_\diamond = \max_{\psi_{RA}} \left\|\mathcal{N}^0_{A\to B}(\psi_{RA}) - \mathcal{N}^1_{A\to B}(\psi_{RA})\right\|_1, \quad (2)$$

where the optimization is over every pure bipartite state ρ_{RA} and the system R is isomorphic to the channel input system A.